

LINEAR PROPERTIES OF BANACH SPACES AND LOW DISTORTION EMBEDDINGS OF METRIC GRAPHS

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Dedicated to the memory of Luis Sánchez González

ABSTRACT. We characterize non-reflexive Banach spaces by a low-distortion (resp. isometric) embeddability of a certain metric graph up to a renorming. Also we study non-linear sufficient conditions for ℓ_1^n being $(1 + \varepsilon)$ -isomorphic to a subspace of a Banach space X .

1. INTRODUCTION

In [10], L. Sánchez and the author have shown the following characterization.

Theorem 1. *There is a bounded countable metric graph M_{ℓ_1} with the following properties.*

a) *If M_{ℓ_1} Lipschitz-embeds into a Banach space X with distortion $D < 2$ (denoted $M \xhookrightarrow{D} X$), then X contains an isomorphic copy of ℓ_1 .*

b) *Conversely, if X contains an isomorphic copy of ℓ_1 then there is an equivalent norm $|\cdot|$ on X such that M_{ℓ_1} embeds isometrically into $(X, |\cdot|)$.*

In this article we modify the methods used to prove this theorem to obtain some further results. Namely, the aim of Section 2 is to present a similar bounded countable metric graph M_R which satisfies the above theorem with the property “ X contains an isomorphic copy of ℓ_1 ” replaced by the property “ X is non-reflexive” (Theorem 3). In Section 3 we briefly discuss the importance of the renorming in these theorems and answer some quantitative questions left open in [10].

In Section 4 we establish a local version of Theorem 1 a), i.e. a theorem where M_{ℓ_1} is replaced by a finite metric space and ℓ_1 is replaced by a finite dimensional ℓ_1^n . In fact, using an ultraproduct argument one can get quite immediately from Theorem 1 a) the following.

Theorem 2. *Let (M_n) be an increasing sequence of finite subsets of the metric space M_{ℓ_1} such that $M_{\ell_1} = \bigcup M_n$. Then for every $\varepsilon > 0$, $D \in [1, 2)$ and every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that if $M_k \xhookrightarrow{D} X$ then $\ell_1^n \subset X$ with linear distortion less than $1 + \varepsilon$.*

The downside is that this theorem nor its proof do not provide any information about the dependence of k on n . The goal of Section 4 is dispensing of the ultraproduct argument in order to study the quantitative dependence between the size of the metric space M_k and the dimension of ℓ_1^n (Theorem 8).

Finally in Section 5 we will prove Theorem 2 and indicate some directions for possible future research.

Throughout the paper we will need the following notions and notation. A mapping $f : M \rightarrow N$ between metric spaces (M, d) and (N, ρ) is called *Lipschitz embedding* if there are constants

$C_1, C_2 > 0$ such that $C_1 d(x, y) \leq \rho(f(x), f(y)) \leq C_2 d(x, y)$ for all $x, y \in M$. The distortion $\text{dist}(f)$ of f is defined as $\inf \frac{C_2}{C_1}$ where the infimum is taken over all constants C_1, C_2 which satisfy the above inequality. We say that M Lipschitz embeds (embeds for brevity) into N with distortion D (in short $M \hookrightarrow_D N$) if there exists a Lipschitz embedding $f : M \rightarrow N$ with $\text{dist}(f) \leq D$. In this case, if the target space N is a Banach space, we may always assume (by taking $C_1^{-1}f$) that $C_1 = 1$.

If X is a Banach space, we will denote by B_X (resp. S_X) the closed unit ball (resp. unit sphere) of X .

For integers $m \leq n$ we denote $\llbracket m, n \rrbracket = [m, n] \cap \mathbb{N}$ and $\llbracket m, \infty \rrbracket = [m, \infty) \cap \mathbb{N}$. For a set S and an integer $n \in \mathbb{N}$ we put $\binom{S}{n} = \{A \subset S : |A| = n\}$, the n -element subsets of S . If $x \in \mathbb{R}$ we will denote by $\lceil x \rceil$ the smallest integer $n \geq x$.

2. LOW-DISTORTION CHARACTERIZATION OF REFLEXIVITY

Let $M_R = \{\mathbf{0}\} \cup \mathbb{N} \cup F$ where $F = \{\llbracket 1, n \rrbracket : n \in \mathbb{N}\} \cup \{\llbracket n, \infty \rrbracket : n \in \mathbb{N}\}$. We introduce on M_R a graph structure: the edges are couples of the form $\{\mathbf{0}, a\}$, $a \in \mathbb{N}$, or $\{a, A\}$, $a \in \mathbb{N}$, $A \in F$ and $a \in F$. Finally, we equip M_R with the shortest path distance.

Theorem 3. *a) Let X be a Banach space and $D \in [1, 2)$. If $M_R \hookrightarrow_D X$ then X is non-reflexive.*

b) Conversely, if X is non-reflexive then there is an equivalent norm $|\cdot|$ on X such that M_R embeds isometrically into $(X, |\cdot|)$.

Proof. a) Assume that $f : M \rightarrow X$ satisfies $f(\mathbf{0}) = 0$ and

$$d(x, y) \leq \|f(x) - f(y)\| \leq Dd(x, y)$$

for some $D < 2$ and all $x, y \in M_R$. Then, each $x^* \in B_X$ that norms $f(\llbracket 1, n \rrbracket) - f(\llbracket n+1, \infty \rrbracket)$ satisfies

$$\inf_{k \leq n} \langle x^*, f(k) \rangle - \sup_{k > n} \langle x^*, f(k) \rangle \geq 4 - 2D$$

Hence $(f(n))_{n \in \mathbb{N}} \subset DB_X$ satisfies $\text{dist}(\overline{\text{co}} \{f(i)\}_{i=1}^n, \overline{\text{co}} \{f(i)\}_{i=n+1}^\infty) \geq 4 - 2D$ for every $n \in \mathbb{N}$. By a well known lemma of James [3, page 51], X is not reflexive.

b) Let us first observe that M_R embeds isometrically into $(c, \|\cdot\|_\infty)$, the space of convergent sequences. We define $\Phi : M_R \rightarrow c$ by

$$\begin{aligned} \Phi(\mathbf{0}) &= 0 \\ \Phi(\llbracket 1, n \rrbracket) &= 2\mathbf{1}_{\llbracket n+1, \infty \rrbracket} \\ \Phi(\llbracket n, \infty \rrbracket) &= -2\mathbf{1}_{\llbracket 1, n \rrbracket} \\ \Phi(n) &= -\mathbf{1}_{\llbracket 1, n \rrbracket} + \mathbf{1}_{\llbracket n+1, \infty \rrbracket} \end{aligned}$$

Then Φ is an isometric embedding. Indeed, $\llbracket 1, n \rrbracket \cap \llbracket m, \infty \rrbracket = \emptyset$ iff $m \geq n+1$. In this case the supports of $\Phi(\llbracket 1, n \rrbracket)$ and $\Phi(\llbracket m, \infty \rrbracket)$ intersect and we have $\|\Phi(\llbracket 1, n \rrbracket) - \Phi(\llbracket m, \infty \rrbracket)\|_\infty = 4$. Otherwise the supports do not intersect and we have $\|\Phi(I) - \Phi(J)\|_\infty = 2$. For the possible distances between $\Phi(\llbracket 1, n \rrbracket)$ and $\Phi(m)$, resp. $\Phi(\llbracket n, \infty \rrbracket)$ and $\Phi(m)$, consult Figure 1.

Now, let Y be a one-codimensional subspace of X . Clearly, Y is not reflexive. Let $\theta \in (0, 1)$. By the proof of James lemma (see [3, page 52]), there exist $F \in B_{Y^{**}}$ and sequences $(x_n) \subset S_Y$,

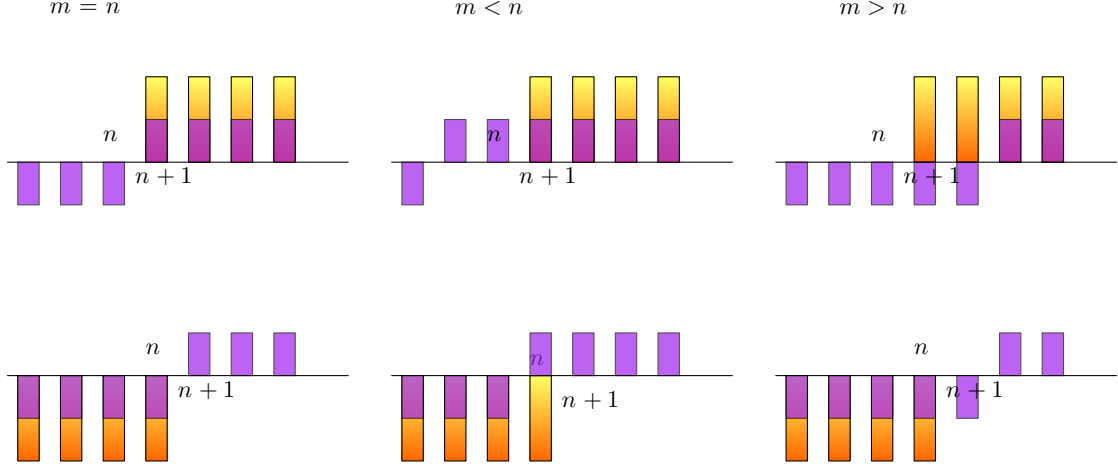


FIGURE 1. The purple bars correspond to $\Phi(m)$, the orange bars correspond to $\Phi([1, n])$ in the first line and to $\Phi([n, \infty[)$ in the second line.

$(x_n^*) \subset S_{Y^*}$ such that

$$\begin{aligned} F(x_n^*) &= \theta \text{ for all } n \in \mathbb{N}, \\ x_n^*(x_k) &= \theta \text{ for all } n \leq k, \\ x_n^*(x_k) &= 0 \text{ for all } n > k. \end{aligned}$$

Observe that X is isomorphic to $Z := Y \oplus \text{span}\{F\} \subset Y^{**}$. We are going to renorm Z and embed M isometrically into this renorming.

Let $y_n \in 2B_Z$ such that $x_n^*(y_n) = 1$. Let $C = \frac{\theta}{12}$. We define, for every $n \in \mathbb{N}$ and every $x \in Z$,

$$\|x\|_n := \max\{|x_n^*(x)|, C\|x - x_n^*(x)y_n\|\}.$$

Then

$$\frac{C}{4}\|x\| \leq \|x\|_n \leq \|x\|$$

for every $n \in \mathbb{N}$ and so $|x| := \sup_n \|x\|_n$ defines an equivalent norm on Z . We define an embedding $f : M \rightarrow (Z, |\cdot|)$ as follows

$$\begin{aligned} f(\mathbf{0}) &= 0 \\ f([1, n]) &= 2(F - x_n) \\ f([n, \infty[) &= -2x_n \\ f(n) &= F - 2x_n. \end{aligned}$$

When evaluated against the functionals (x_k^*) this embedding reminds the embedding $\Phi : M \rightarrow c$. Indeed, one can check easily that for every $a \in M$ and each $k \in \mathbb{N}$ we have $\langle x_k^*, f(a) \rangle = \theta \Phi(a)(k)$. Moreover $C\|x - x_n^*(x)y_n\| \leq \theta$ for all $n \in \mathbb{N}$ whenever $x = f(a) - f(b)$. It follows that

$$|f(a) - f(b)| = \sup_{n \in \mathbb{N}} |x_n^*(f(a) - f(b))| = \theta \|\Phi(a) - \Phi(b)\|_\infty = \theta d(a, b)$$

for every $a, b \in M$. Thus $g = f/\theta$ is the desired embedding. □

3. THE ROLE OF STABILITY

We recall that a metric space (M, d) is *stable* if for all bounded sequences $(x_n), (y_n) \subset M$ and for all non-principal ultrafilters \mathcal{U}, \mathcal{V} we have

$$\lim_{n, \mathcal{U}} \lim_{m, \mathcal{V}} d(x_n, y_m) = \lim_{m, \mathcal{V}} \lim_{n, \mathcal{U}} d(x_n, y_m).$$

The space ℓ_1 is stable, see [4, page 212].

The next proposition shows at once that in general we need to pass to a renorming in both theorems 1 b) and 3 b).

Proposition 4. *Let $C > 1$. Let M be uniformly discrete and bounded metric space with the property that $M \hookrightarrow_D X$, $D < C$ only if X is non-reflexive. Then $M \hookrightarrow_D \ell_1 \Rightarrow D \geq C$.*

Proof. Let $\Phi : M \hookrightarrow_D \ell_1$, $D < C$. Then $\Phi(M)$ is uniformly discrete, bounded and stable (see [4]).

Baudier and Lancien [2] have shown that stable metric spaces nearly isometrically embed into the class of reflexive spaces. In the uniformly discrete and bounded case that means that there exists a reflexive space X such that $\Phi(M) \hookrightarrow_1 X$, thus $M \hookrightarrow_D X$ which is impossible. \square

Remark 5. If $(x_n) \subset B_X$ is a 1-separated sequence, then any bijection $f : M_R \rightarrow \{x_n\}$ satisfies $\text{dist}(f) \leq 8$. Thus M_R embeds with distortion 8 into any infinite-dimensional Banach space. We do not know whether M_R embeds into some reflexive space X with distortion 2. In other words, we do not know whether the constant 2 in Theorem 3 a) is optimal.

On the other hand, let $\rho : M_R \times M_R \rightarrow \mathbb{R}$ be defined by $\rho(\mathbf{0}, n) = \frac{2}{3}$, $\rho(\mathbf{0}, A) = \frac{4}{3}$, $\rho(n, A) = 2$, $\rho(n, m) = \frac{4}{3}$ and $\rho(A, B) = \frac{8}{3}$ for all $n \neq m \in \mathbb{N}$ and all $A \neq B \in F \subset M_R$. Then ρ is a stable metric on M_R and $\text{dist}(Id) = 3$ for the identity $Id : (M_R, d) \rightarrow (M_R, \rho)$. To see the stability consider the following isometric embedding $g : (M_R, \rho) \rightarrow \ell_1(M_R)$ defined by $g(\mathbf{0}) = 0$, $g(n) = \frac{2}{3}e_n$, for every $n \in \mathbb{N}$, and $g(A) = \frac{4}{3}e_A$, for every $A \in F$. It follows, using again [2], that there is a reflexive Banach space X such that $M_R \hookrightarrow_3 X$.

Finally let us mention that the method using [2] does not work for distortions less than 3 as it follows from the next lemma that M_R does not embed into a stable metric space with distortion less than 3.

Lemma 6. *Let (M, d) be a metric space containing sequences (x_n) and (y_n) such that*

$$\lim_{n, \mathcal{U}} \lim_{m, \mathcal{V}} d(x_n, y_m) \geq C \lim_{m, \mathcal{V}} \lim_{n, \mathcal{U}} d(x_n, y_m).$$

Then M does not embed into any stable metric space with distortion $D < C$.

Proof. Let $D < C$ and assume that

$$sd(x, y) \leq d(f(x), f(y)) \leq sDd(x, y)$$

for some $f : M \rightarrow N$ and some stable metric space N . Then

$$\begin{aligned} s \lim_{n, \mathcal{U}} \lim_{m, \mathcal{V}} d(x_n, y_m) &\leq \lim_{n, \mathcal{U}} \lim_{m, \mathcal{V}} d(f(x_n), f(y_m)) = \lim_{m, \mathcal{V}} \lim_{n, \mathcal{U}} d(f(x_n), f(y_m)) \\ &\leq sD \lim_{m, \mathcal{V}} \lim_{n, \mathcal{U}} d(x_n, y_m) < sC \lim_{m, \mathcal{V}} \lim_{n, \mathcal{U}} d(x_n, y_m) \leq s \lim_{n, \mathcal{U}} \lim_{m, \mathcal{V}} d(x_n, y_m) \end{aligned}$$

which is impossible. \square

Remark 7. We recall the definition of the space M_{ℓ_1} . The vertex set is $M_{\ell_1} = \{\mathbf{0}\} \cup \mathbb{N} \cup F$ with $F = \{A \subset \mathbb{N} : 1 \leq |A| < \infty\}$. A pair $\{a, b\}$ is an edge either if $a = \mathbf{0}$ and $b \in \mathbb{N}$, or if $a \in \mathbb{N}$, $b \in F$ and $a \in b$. The space M_{ℓ_1} is equipped with the shortest path metric.

In [10], we asked for the best constant D such that $M_{\ell_1} \xrightarrow[D]{} \ell_1$ and also for the value of $d_{BM}(\ell_1, \mathcal{F}(M_{\ell_1}))$ where $\mathcal{F}(M_{\ell_1})$ is the Lipschitz free space of M_{ℓ_1} .

First, using the above lemma, one can easily see that M_{ℓ_1} does not embed with distortion less than 3 into any stable space. Second, if we define $\rho : M_{\ell_1} \times M_{\ell_1} \rightarrow \mathbb{R}$ as in Remark 5, we have that $\text{dist}(Id) = 3$ for the identity map $Id : (M_{\ell_1}, d) \rightarrow (M_{\ell_1}, \rho)$. Defining the isometric embedding $g : (M_{\ell_1}, \rho) \rightarrow \ell_1$ as in Remark 5, this already gives that $D = 3$.

But it also follows from the theory of Lipschitz free spaces that $\|\widehat{Id}\| \|\widehat{Id}^{-1}\| = 3$ where $\widehat{Id} : \mathcal{F}(M_{\ell_1}, d), \mathcal{F}(M_{\ell_1}, \rho)$ is the unique linear extension of Id . Moreover, $\mathcal{F}(M_{\ell_1}, \rho) \equiv \ell_1$. This can be seen by noticing that $\mathcal{F}(M_{\ell_1}, \rho)$ is isometric to a negligible subset of an \mathbb{R} -tree which contains all the branching points, and applying [6]. Thus $d_{BM}(\ell_1, \mathcal{F}(M_{\ell_1})) = 3$.

4. LOW-DISTORTION REPRESENTATION OF ℓ_1^n

In this section we state and prove a quantitative version of Theorem 2 for a particular choice of spaces $M_n \subset M_{\ell_1}$. Having in mind the definition of the space M_{ℓ_1} (see Remark 7), the most natural choice of the spaces M_n seems to be the following. We put $M_n = \{\mathbf{0}\} \cup \llbracket 1, n \rrbracket \cup F_n$ where $F_n = 2^{\llbracket 1, n \rrbracket} \setminus \{\emptyset\}$. The graph structure and the metric are induced by the space M_{ℓ_1} .

The main result of this section follows.

Theorem 8. *a) Let $D \in [1, \frac{4}{3})$ and $n \in \mathbb{N}$. Then $M_n \xrightarrow[D]{} X$ implies that ℓ_1^n is $\frac{D}{4-3D}$ -isomorphic to a subspace of X .*

b) Let $D \in [1, 2)$. For every $\alpha \in (0, 1)$ there exists $\eta = \eta(\alpha, D) \in (0, 1)$ such that $M_k \xrightarrow[D]{} X$

implies that $\ell_1^{\lceil \eta k \rceil}$ is $\frac{2D}{2-D}$ -isomorphic to a subspace of X whenever $k > \frac{\log_2(\frac{2D}{2-D})}{1-\alpha}$.

The isomorphism constant can be arbitrarily reduced, at the cost of augmenting the size of the metric space, by virtue of the following finite version of James's ℓ_1 -distortion theorem, see Proposition 30.5 in [12]: *If X contains a b^2 -isomorphic copy of $\ell_1^{m^2}$, then X contains a b -isomorphic copy of ℓ_1^m .*

For example, in the case **a)**, we get: *If $D < \frac{4}{3}$ and $w \geq -\log_2 \left(\frac{\log_2(1+\varepsilon)}{\log_2(\frac{2D}{4-3D})} \right)$, then $M_{n^{2^w}} \xrightarrow[D]{} X$ implies that ℓ_1^n is $(1+\varepsilon)$ -isomorphic to a subspace of X .*

In order to prove Theorem 8 we will need the following lemma, which is a finite-dimensional version of Proposition 4 in [11].

Lemma 9. *Let S be a set, $K > 0$ and let $(f_i)_{i=1}^n \subset KB_{\ell_\infty(S)}$. Assume that there are $\delta > 0$ and $r \in \mathbb{R}$ such that these functions satisfy that for every $A \subset \llbracket 1, n \rrbracket$ there is $s \in S$ for which*

$$f_j(s) \leq r < r + \delta \leq f_i(s)$$

for all $i \in A$ and for all $j \in \llbracket 1, n \rrbracket \setminus A$. Then $(x_i)_{i=1}^n$ is $\frac{2K}{\delta}$ -equivalent to the unit vector basis of ℓ_1^n .

Proof. Let $(c_i)_{i=1}^n$ be a set of real coefficients. We put $A = \{i \in \llbracket 1, n \rrbracket : c_i \geq 0\}$. Let $s, t \in S$ satisfy

$$\begin{aligned} f_j(s) &\leq r < r + \delta \leq f_i(s) \\ f_i(t) &\leq r < r + \delta \leq f_j(t) \end{aligned}$$

for all $i \in A$ and $j \in \llbracket 1, n \rrbracket \setminus A$. Then $f_i(s) - f_i(t) \geq r + \delta - r = \delta$ if $i \in A$ and $f_j(s) - f_j(t) \leq r - (r + \delta) = -\delta$ if $j \notin A$. It follows that

$$\sum_{1 \leq i \leq n} c_i(f_i(s) - f_i(t)) \geq \sum_{i \in A} c_i \delta - \sum_{i \notin A} c_i \delta = \delta \sum_{1 \leq i \leq n} |c_i|$$

Hence $\max \{|\sum_{i=1}^n c_i f_i(s)|, |\sum_{i=1}^n c_i f_i(t)|\} \geq \frac{\delta}{2} \sum |c_i|$. Thus $\frac{\delta}{2} \sum_{i=1}^n |c_i| \leq \|\sum_{i=1}^n c_i f_i\|_\infty \leq K \sum_{i=1}^n |c_i|$. \square

Proof of Theorem 8 a). Assume that $f : M_n \rightarrow X$ satisfies $f(\mathbf{0}) = 0$ and

$$d(x, y) \leq \|f(x) - f(y)\| \leq Dd(x, y)$$

for some $D < \frac{4}{3}$ and all $x, y \in M_n$. We put

$$X_{a,b} = \{x^* \in B_{X^*} : \langle x^*, f(a) \rangle \geq 4 - 3D, \langle x^*, f(b) \rangle \leq -4 + 3D\}.$$

We claim that for every $A, B \subset \llbracket 1, n \rrbracket$ such that $A \cap B = \emptyset$ and $A \cup B \neq \emptyset$, we have

$$\bigcap_{a \in A, b \in B} X_{a,b} \neq \emptyset.$$

Indeed, in the case $A \neq \emptyset \neq B$ we take $x^* \in K$ such that $\langle x^*, f(A) - f(B) \rangle = \|f(A) - f(B)\| \geq 4$. Then for any $a \in A$ we have

$$\begin{aligned} \langle x^*, f(a) - f(\mathbf{0}) \rangle &= \langle x^*, f(A) - f(B) \rangle - \langle x^*, f(A) - f(a) \rangle - \langle x^*, f(\mathbf{0}) - f(B) \rangle \\ &\geq 4 - 3D, \end{aligned}$$

and by the same argument we get $\langle x^*, f(b) \rangle \leq -4 + 3D$. Hence $x^* \in \bigcap_{a \in A, b \in B} X_{a,b}$. In the case

when $B = \emptyset$ we get an $x^* \in B_{X^*}$ such that for all $a \in A$ we have $\langle x^*, f(a) \rangle \geq 2 - D \geq 4 - 3D$. Lemma 9 now implies that $(f(i))_{i=1}^n$ is $\frac{D}{4-3D}$ -equivalent to the unit vector basis of ℓ_1^n . \square

The proof of Theorem 8 b) will depend on the following well known lemma.

Lemma 10. *For every $\alpha \in (0, 1]$ there is $\eta = \eta(\alpha) > 0$ such that for each $k \in \mathbb{N}$ and each $\mathcal{S} \subset 2^{\llbracket 1, k \rrbracket}$, the estimate $|\mathcal{S}| \geq 2^{\alpha k}$ implies that there exists $H \in \binom{\llbracket 1, k \rrbracket}{\eta k}$ such that $\{A \cap H : A \in \mathcal{S}\} = 2^H$.*

The proof of this lemma is in turn based on the combination of the following two lemmas, see [8, pages 402-403] or [9, page 11],

Lemma 11 (Sauer, Shelah, and Vapnik and Červonenkis). *Let $\mathcal{S} \subset 2^{\llbracket 1, k \rrbracket}$ such that $|\mathcal{S}| > \sum_{i=0}^{m-1} \binom{k}{i}$ for some $m \leq k$. Then there is $H \in \binom{\llbracket 1, k \rrbracket}{m}$ such that $\{A \cap H : A \in \mathcal{S}\} = 2^H$.*

Lemma 12. *For every $1 \leq m \leq k$ one has*

$$\sum_{i=0}^m \binom{k}{i} \leq \left(\frac{ek}{m} \right)^m.$$

Proof of Lemma 10. Let $\eta > 0$ satisfy $2^\alpha > \left(\frac{e}{\eta} \right)^\eta$. And $m = \lceil \eta k \rceil$. Then, using $\eta < 1$ and Lemma 12,

$$|\mathcal{S}| \geq 2^{\alpha k} > \left(\frac{ek}{\eta k} \right)^{\eta k} \geq \left(\frac{ek}{m-1} \right)^{m-1} \geq \sum_{i=0}^{m-1} \binom{k}{i}$$

So we get the existence of H by Lemma 11. □

Proof of Theorem 8 b). Given $D < 2$, let $c = \lceil \frac{2D}{2-D} \rceil - 1$. Let $k > \frac{\log_2(c-1)}{1-\alpha}$. Assume that $f : M_k \rightarrow X$ satisfies $f(\mathbf{0}) = 0$ and

$$d(x, y) \leq \|f(x) - f(y)\| \leq Dd(x, y)$$

for all $x, y \in M_k$. To simplify notation we will denote $x' := f(x)$ for each $x \in M_k$. Let $r_j := -D + j(D-2)$ for $j \in \llbracket 1, c \rrbracket$. Then any closed interval of length $4-2D$ which is contained in $[-D, D]$ contains at least two different points r_j . We claim that for every $A \in 2^{\llbracket 1, k \rrbracket}$ there are $j \in \llbracket 1, c-1 \rrbracket$ and $x^* \in B_{X^*}$ such that

$$\langle x^*, b' \rangle \leq r_j < r_{j+1} \leq \langle x^*, a' \rangle$$

for all $a \in A$ and all $b \in B := \llbracket 1, k \rrbracket \setminus A$. Indeed, we take $x^* \in B_{X^*}$ such that

$$\begin{aligned} \langle x^*, A' - B' \rangle &= \|A' - B'\| \text{ if } A \neq \emptyset \neq B \\ \langle x^*, A' \rangle &= \|A'\| \text{ if } B = \emptyset \\ \langle x^*, -B' \rangle &= \|B'\| \text{ if } A = \emptyset \end{aligned}$$

In the first case we get for all $a \in A$ and all $b \in B$ that $\langle x^*, a' - b' \rangle \geq 4 - 2D$. Moreover $\langle x^*, a' \rangle, \langle x^*, b' \rangle \in [-D, D]$ so the claim follows by the choice of (r_j) . In the second case we get for all $a \in A$ that $\langle x^*, a' \rangle \geq 2 - D$, and in the last case we get for all $b \in B$ that $\langle x^*, b' \rangle \leq D - 2$. In both cases we use that the interval $[D-2, 2-D]$ contains at least two different r_j to finish the proof of the claim.

Let us choose for every $A \subset \llbracket 1, k \rrbracket$ one such j which we will denote j_A . By the pigeonhole principle, there is some $j \in \llbracket 1, c-1 \rrbracket$ such that $|\mathcal{S}| \geq \frac{2^k}{c-1}$ for $\mathcal{S} = \{A \in 2^{\llbracket 1, k \rrbracket} : j_A = j\}$. By the choice of k we have $\frac{2^k}{c-1} \geq 2^{\alpha k}$. Let $\eta \in (0, 1)$ and $H \in \binom{\llbracket 1, k \rrbracket}{\lceil \eta m \rceil}$ be as in Lemma 10. Applying Lemma 9 we can see that $(f(i))_{i \in H}$ is $\frac{2D}{2-D}$ -equivalent to the unit vector basis of $\ell_1^{\lceil \eta k \rceil}$. □

5. ULTRAPRODUCT TECHNIQUES FOR LOW DISTORTION REPRESENTATION

Let us first see the proof of Theorem 2.

Proof of Theorem 2. Suppose that the assertion is not true for some $\varepsilon > 0$, $D \in [1, 2)$ and $n \in \mathbb{N}$. Then for every k there is X_k such that $M_k \hookrightarrow_D X_k$ and ℓ_1^n is not $(1 + \varepsilon)$ -isomorphic to a subspace of X_k . Let $X = \prod X_k$ be an ultraproduct along some free ultrafilter on \mathbb{N} . Then $M_{\ell_1} \hookrightarrow_D X$ and so, by Theorem 1 a), ℓ_1 embeds into X linearly and does so arbitrarily well (by

James's ℓ_1 -distortion theorem, see e.g. [1] . Therefore ℓ_1^n embeds into X arbitrarily well and in particular it must be $(1 + \frac{\varepsilon}{2})$ -isomorphic to a subspace of some X_k . Contradiction. \square

Let us mention the following folklore result which according to G. Lancien goes back to G. Schechtmann.

Theorem 13. *Let X be a Banach space such that $\dim X < \infty$. Then for every $D \geq 1$ and $\varepsilon > 0$ there is a finite set $F \subset X$ such that for any given Banach space Y the fact $F \xrightarrow[D]{} Y$ implies that X is $(D + \varepsilon)$ -isomorphic to a subspace of Y .*

Proof. Let $(F_n) \subset 2^X$ be any increasing family of finite sets such that $\bigcup F_n = X$. Let us assume that there are $D > 1$ and $\varepsilon > 0$ such that for every $k \in \mathbb{N}$ there is a Banach space Y_k and an embedding $f_k : F_k \xrightarrow[D]{} Y_k$ but X is not $(D + \varepsilon)$ -isomorphic to any subspace of Y_k . Then $\Phi(x) := [(f_n(x))_n]$ for $x \in \bigcup F_n$ and extended by continuity to the whole X is an embedding of X into $\prod_{\mathcal{U}} Y_k$ with distortion D . By the theorem of Heinrich and Mankiewicz [4, Theorem 7.9] X linearly embeds into $(\prod_{\mathcal{U}} Y_k)^*$ with distortion D . By local reflexivity X embeds linearly into $\prod_{\mathcal{U}} Y_k$ with distortion $D + \varepsilon/3$ and therefore X embeds into some Y_k linearly with distortion $D + 2\varepsilon/3$ which is a contradiction. \square

Remark 14. 1) Clearly the above theorem is qualitatively better than Theorem 2 as it works for any space X and without any restriction on the distortion. Also the set F is isometrically in X . On the other hand, any other information on the nature of F is completely inaccessible. It could be interesting to give a concrete example of such sets for a given finite dimensional space X .

2) When $X = \ell_p^n$ with $1 \leq p \leq 2$, one can say more. Let C_p^n be the “ n -cube” equipped with the ℓ_p -norm. We recall a result of Bourgain, Milman and Wolffson [5, page 297] which says: Let $1 \leq p \leq 2$. Assume that there exists $D > 0$ such that $C_p^n \xrightarrow[D]{} Y$ for every n . Then for every $\varepsilon > 0$ and $n \in \mathbb{N}$ there is a subspace of Y which is $(1 + \varepsilon)$ -isomorphic to ℓ_p^n .

Essentially the same proof as the one of Theorem 2 gives therefore the following. Let $1 \leq p \leq 2$. Then for every $\varepsilon > 0$, $D > 0$ and $n \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that for every Banach space Y we have that $C_p^k \xrightarrow[D]{} Y$ implies that ℓ_p^n is $(1 + \varepsilon)$ -isomorphic to a subspace of Y .

The dependence of k on n does not seem to follow from the proof in [5].

ACKNOWLEDGEMENTS

I'm grateful to Gilles Godefroy for pointing out Theorem 2 to me. I am also grateful to Beata Randrianantoanina and Gilles Lancien for inspiring discussions. A part of this paper was written during the conference “Banach spaces and their applications in analysis” at CIRM. I would like to thank the organizers for inviting me, and the CIRM for its hospitality.

REFERENCES

- [1] F. Albiac and N. Kalton, *Topics in Banach space theory*, Graduate Texts in Mathematics, vol. 233, Springer, New York, 2006.
- [2] F. Baudier and G. Lancien, *Tight embeddability of proper and stable metric spaces*, Anal. Geom. Metr. Spaces 3 (2015), 140-156.

- [3] B. Beauzamy, *Introduction to Banach Spaces And Their Geometry*, North-Holland mathematics studies 68, 1982.
- [4] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis*, AMS Colloquium publications, Volume 48.
- [5] J. Bourgain, V. Milman and H. Wolfson, *On type of metric spaces*, Trans. Amer. Math. Soc. 294, No. 1, (1986), 295–317.
- [6] A. Godard, *Tree metrics and their Lipschitz-free spaces*, Proc. Amer. Math. Soc. 138 (2010), no. 12, 4311–4320.
- [7] G. Godefroy and N. J. Kalton, *Lipschitz-free Banach spaces*, Studia Math., **159**, (2003), 121–141.
- [8] D. Li and H. Queffélec, *Introduction à l'étude des espaces de Banach: Analyse et probabilités*, SMF, 2004.
- [9] A. Pajor, *Sous-espaces ℓ_1^n des espaces de Banach*, Travaux en cours, Hermann, Paris, 1985.
- [10] A. Procházka and L. Sánchez, *Low distortion embeddings into Asplund Banach spaces*, arXiv:1311.4584.
- [11] H. Rosenthal, *A characterization of Banach spaces containing ℓ_1* , Proc. Nat. Acad. Sci. USA, Vol. 71, No. 6, 2411–2413, June 1974.
- [12] N. Tomczak-Jaegermann, *Banach-Mazur distances and finite-dimensional operator ideals*, Pitman monographs and surveys in pure and applied mathematics; 38, 1989.

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